Lecture #22

CS 170

Spring 2021

Randomized Algorithms

- When we can't be both fast and correct, settle for
- Correct, and probably fast (Las Vegas)

- Fast, and probably correct (Monte Carlo)

Quick Review of Probability (CS70)

- Random variable X takes values $x_1, x_2, \ldots$ with probabilities $P(X = x_i) = p_i \geq 0$, $\sum_i p_i = 1$
  - Roll fair die, $X \in \{1 \ldots 6\}$ each with $p_i = \frac{1}{6}$
- Expectation of X = "average value of X"
  $= \mathbb{E}(X) = \sum_i x_i p_i$
  - Roll fair die, $\mathbb{E}(X) = \frac{1}{6}(1+2+\cdots+6) = 3.5$
- If X and Y are random variables, $\mathbb{E}(aX+bY) = a\,\mathbb{E}(X) + b\,\mathbb{E}(Y)$
- Markov's Inequality: If $X \geq 0$ then

Randomized Quicksort of $A(1:n)$

- Assume w.l.o.g. that all $A(i)$ distinct
  - Else lexicographic: $(A(i),i) < (A(j),j)$ if $A(i)<A(j)$ elseif $i<j$

Quicksort($A(1:n)$):
  if $n=1$ return $A(1)$, else
    pick uniformly random pivot $i \in \{1,\dots,n\}$
    $L \leftarrow \{i : A(i) < A(\text{pivot})\}$
    $R \leftarrow \{i : A(i) > A(\text{pivot})\}$
    return (Quicksort($A(L)$), A(pivot),
        Quicksort($A(R)$))

Worst case:

Best case:

Hope:

Proof that $\mathbb{E}T(n) = O(n \log n)$ (1/2)

- $T(n) = \Theta(\#\text{ comparisons})$
- $X_{ij} = 1$ if i^{th} smallest entry compared to j^{th} smallest, else 0
  -
- Each X_{ij} is a random variable so
- How is X_{ij} determined?
  - Let $a_1 < a_2 < \cdots < a_n$ be sorted array
  -
  -
  -

Proof that $\mathbb{E}T(n) = O(n \log n)$ (2/2)

- $T(n) = \Theta(\#\text{ comparisons})$
- $X_{ij} = 1$ if i^{th} smallest entry compared to j^{th} smallest, else 0
  - $\#\text{ comparisons} = \sum_{i<j} X_{ij} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}$
- Each X_{ij} is a random variable so

  $\mathbb{E}(\#\text{ comparisons}) =$

- Markov inequality:

Freivald's Algorithm (1/2)

- Given $n \times n$ matrices A, B, C, test whether $C = A \cdot B$ faster than multiplying $A \cdot B$

- Intuition:

- Thm:

- Cor:

Freivald's Algorithm

- Thm: if $x \in \{0,1\}^n$ chosen with each entry x_i independent, $P(x_i = 0) = \frac{1}{2} = P(x_i = 1)$, and $C \neq A \cdot B$, then $P(Cx \neq ABx) \geq \frac{1}{2}$
- Proof:

Karger's Global Mincut Algorithm (1/6)

- Def: A cut of an undirected graph $G(V,E)$ is a partition $V = S \cup \bar{S}$, $S \cap \bar{S} = \phi$, $S \neq \phi$, $\bar{S} \neq \phi$
- Def: Size of a cut = # edges connecting $S, \bar{S}$
  $= |E \cap (S \times \bar{S})|$
- Def: Global Mincut (GMC) $= (S, \bar{S})$ minimizing size

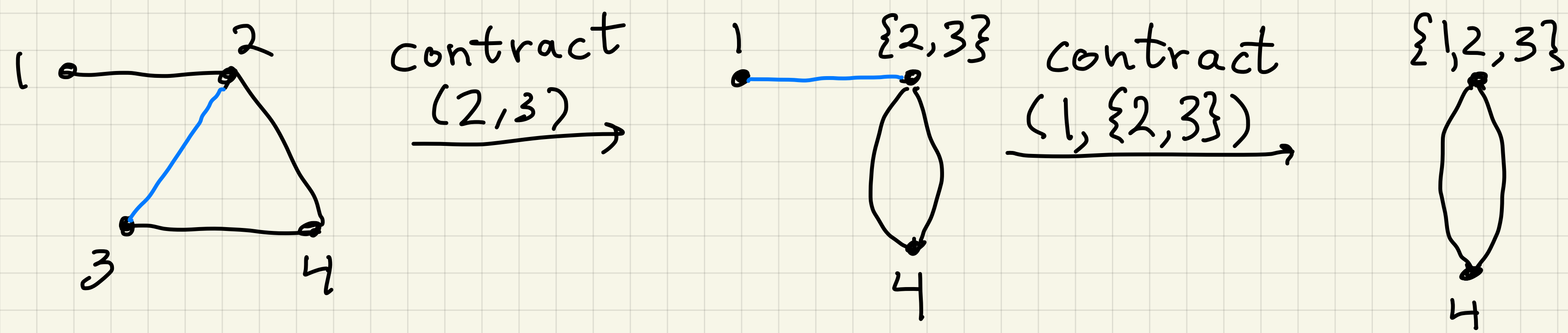
Karger's Global Mincut Algorithm (2/6)

- Def: Given $G(V, E)$, contract(e), $e=(u,v)$

  means 1)

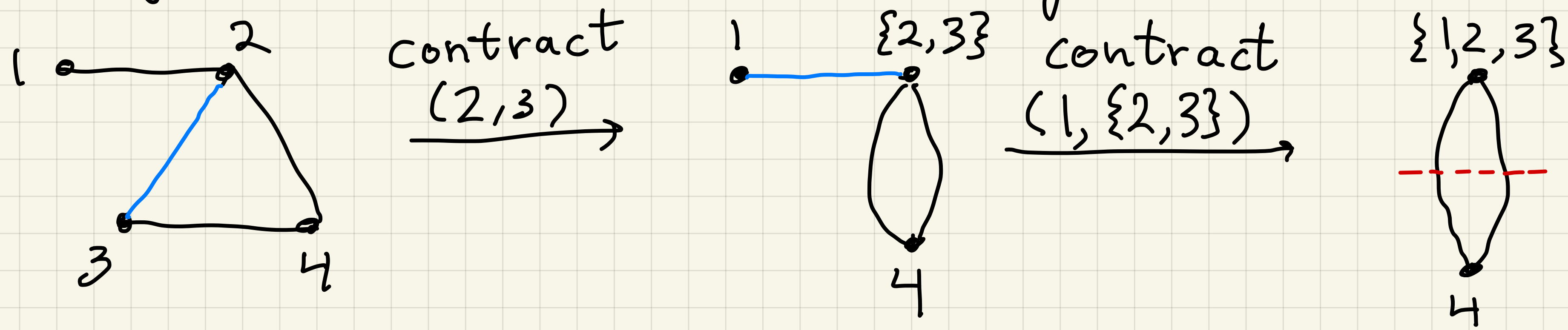
  2)

  3)



- Karger's Algorithm:

Karger's Global Mincut Algorithm (3/6)



- Karger's Algorithm:
  for $i = 1$ to $|V|-2$
  pick random edge e, contract(e)
  return cut determined by last 2 vertices

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Karger's Global Mincut Algorithm (4/6)

- Karger's Algorithm:

  for $i = 1$ to $|V| - 2$ ... $n = |V|$ below

  pick random edge e, contract(e)

  return cut determined by last 2 vertices

- Fact: Karger returns GMC $(S, \bar{S})$

  $\Leftrightarrow$ never contracts an edge in GMC

- What is probability that Karger never contracts an edge in GMC?

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Karger's Global Mincut Algorithm (5/6)

- Karger's Algorithm:

```
for i = 1 to |V|-2
    pick random edge e, contract(e)
return cut determined by last 2 vertices
```

- m_i = # edges, k = size of GMC (# edges from S to $\bar{S}$)
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-

Karger's Global Mincut Algorithm (6/6)

- Karger's Algorithm:

```
for i = 1 to |V| - 2
    pick random edge e, contract(e)
return cut determined by last 2 vertices
```

- $P(\text{Karger gets right answer}) \geq 1 / \binom{n}{2}$
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One more "hot topic"

- Lots of current research on randomized linear algebra algorithms
  - Least squares problems $\min_x \|Ax-b\|_2$
  - PCA (Principle Component Analysis)
  - SVD (Singular Value Decomposition)
- see "References for Randomized Algorithms" at people.eecs.berkeley.edu/~demmel/ma221_Fall20
- High level common approach: replace A by RA, R = random matrix, solve using RA instead